Discipline: **Physics**

Subject: **Electromagnetic Theory**

Unit 29:

Lesson/ Module: **Thomson Scattering**

Author (CW): **Prof. V. K. Gupta**

Department/ University: **Department of Physics and Astrophysics,**

- **University of Delhi, New Delhi-110007**

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Learning Objectives:

From this module, a continuation of module 26, students may get to know about the following:

1. *Scattering of incident electromagnetic radiation by a free charged particle, say an electron.*
2. *Scattering of radiation by a system of quasi-free charges and their coherent and incoherent addition in the background of X-ray scattering.*
3. *Scattering by a bound charge including the effect of radiation damping.*
4. *Analysis of the result obtained in different frequencies. Rayleigh scattering and resonance fluorescence.*

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29 Thomson Scattering

29.1 Scattering of radiation

If a plane electromagnetic wave is incident on a free charged particle, the particle will feel a force due to the electromagnetic field of the incident wave. This force will accelerate the charged particle; an accelerated charged particle emits radiation. Thus an electromagnetic wave impinging on a charged particle leads to radiation by the charged particle. The whole process can be viewed as the scattering of the incident wave by a charged particle. It is this phenomenon that we wish to study in this module.

We assume that the intensity of the incident wave is such that the motion of the charged particle induced by it is non-relativistic. The emitted radiation will have the same frequency as that of the incident wave.

The fields produced by a particle of mass m and charge e moving with velocity $\vec{v} = c\vec{\beta}$ and acceleration $\dot{\vec{v}} = c\dot{\vec{\beta}}$ were studied in module 26 and the emitted power in module 27. We recall the expressions for the fields in the non-relativistic limit from module 27:

$$\vec{E}(\vec{x},t) = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right]_{ret} = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{(\hat{n}.\dot{\vec{\beta}})\hat{n}-\dot{\vec{\beta}}}{R}\right]_{ret} \tag{1}$$

$$\vec{B} = \frac{1}{c}[\hat{n}\times\vec{E}]_{ret} \tag{2}$$

Here $\hat{n}$ is the unit vector in the direction of observation. Only the acceleration part of the field is given since that is the one relevant for radiation. This field is in the plane containing $\hat{n}$ and $\dot{\vec{\beta}}$. Since we are assuming that the motion of the particle is non-relativistic, the effect of the magnetic field is negligible, and the acceleration of the charged particle is solely due to the electric field. If the propagation vector of the incident wave is $\vec{k}_0$, the amplitude of the electric field is E_0 and the polarization vector is $\vec{\varepsilon}_0$, then

$$\vec{E}(\vec{x},t) = \vec{\varepsilon}_0 E_0 e^{i(\vec{k}_0.\vec{x}-\omega t)} \tag{3}$$

The acceleration produced in the particle is

$$\dot{\vec{v}}(\vec{x},t) = \vec{\varepsilon}_0 \frac{e}{m} E_0 e^{i(\vec{k}_0.\vec{x}-\omega t)} \tag{4}$$

The expression for power radiated by a non-relativistic particle was found in module 27, and is

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi\varepsilon_0)(4\pi c^3)}\left|\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right|^2 = \frac{e^2}{(4\pi\varepsilon_0)(4\pi c^3)}\left|\hat{\varepsilon}^*.\dot{\vec{v}}\right|^2 \qquad (5)$$

Since the motion is non-relativistic, the charge moves only a small distance during one cycle of oscillation of the incoming wave. As a result, the position vector $\vec{x}$ in equation (4) is constant, and the time average of $\left|\dot{\vec{v}}\right|^2$ is $\frac{1}{2}\mathrm{Re}(\dot{\vec{v}}.\dot{\vec{v}}^*)$. Using this expression in equation (5), the average power per unit solid angle can be written as

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2}{(4\pi\varepsilon_0)(4\pi c^3)}\left|\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right|^2 = \frac{c}{8\pi}\frac{1}{(4\pi\varepsilon_0)}\left(\frac{e^2}{mc^2}\right)^2\left|E_0\right|^2 \qquad (6)$$

We are looking at it as scattering process, scattering of the incident wave by the point particle which acts as a scatterer. Hence we express this result in terms of the scattering cross-section. Now the differential scattering cross-section, $\frac{d\sigma}{d\Omega}$, in the case of radiation is defined as the energy radiated per unit time per unit solid angle divided by the incident energy flux (incident energy per unit area per unit time). The incident energy flux is just the time averaged Poynting vector for the incident plane wave

$$\vec{S} = \frac{1}{2}\varepsilon_0 c\left|\vec{E}\right|^2\hat{n} \qquad (7)$$

Hence the scattering cross-section becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2\left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \qquad (8)$$

Let the incident plane wave be along the z-axis as shown in the figure below. **[See Figure, 14.12, from Jackson Edition 2]** Let the vector $\hat{n}$ make an angle θ with this axis, and choose two directions, $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ in the plane containing ($\hat{n}$, z) and perpendicular to it, respectively. In terms of the unit vectors $\hat{e}_x, \hat{e}_y$ parallel to the x and y axis respectively, we have

$$\begin{aligned}\hat{e}_1 &= \cos\theta(\cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y) - \sin\theta\,\hat{e}_z\\ \hat{e}_2 &= -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y\end{aligned} \qquad (9)$$

If the incident wave is polarized along the x-axis, then $\hat{e}_1.\hat{e}_0 = \cos\theta\cos\phi$ and $\hat{e}_2.\hat{e}_0 = -\sin\phi$. The angular distribution of the cross-section averaged over the final polarization is

$$\left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 = (\cos^2\theta\cos^2\phi + \sin^2\phi) \qquad (10)$$

Similarly, if the incident wave is polarized along the *y*-axis, then $\hat{e}_1.\hat{e}_0 = \cos\theta \sin\phi$ and $\hat{e}_2.\hat{e}_0 = -\cos\phi$. The angular distribution of the cross-section averaged over the final polarization is

$$\left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 = (\cos^2\theta \sin^2\phi + \cos^2\phi) \tag{11}$$

If the incident radiation is unpolarized, the cross-section is the average of the two, and is therefore

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \frac{1}{2}(1+\cos^2\theta) \tag{12}$$

This is the *Thomson formula* for differential scattering cross-section for scattering of radiation by a free charge. It works for scattering of X-rays by electrons or γ-rays for protons. It does not work when the photon momentum and the recoil of the charged particle cannot be neglected. The integral of this over all angles yields the total cross-section

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \tag{13}$$

This is called the *Thomson cross-section*. For electrons this has the value 0.665×10^{-28} m^2. This is as it should be as the cross-section has dimensions of area. The quantity in parentheses has the units of length. For electrons it has the value 2.82×10^{-15} m. This will be the radius of an electron if the total ``mass-energy" of the electron were due to its charge being concentrated in a ball. It is called the *classical electron radius*, an important number to remember. This tells us that even *point* particles have a *finite* scattering cross-section that appears in this limit to be independent of the wavelength of the light scattered.

However, this is not really true if you recall the approximations made - this expression will fail if the wavelength is of the same order as the classical radius. Beyond this pair production becomes a significant process, which is a quantum-mechanical process. In quantum mechanics, if the energy of the incident photon, $\hbar\omega \approx mc^2$, significant momentum is transferred to the electron by the collision and the energy of the scattered photon cannot be equal to the energy of the incident photon. The effect is certainly quantum-mechanical since the very idea of a photon having definite energy and momentum is quantum-mechanical in origin.

As a result of the recoil of the charged particle (electron) by the incident photon, the energy of the photon is less and thereby the wavelength of the radiation is more than that of the incident beam. This leads to what is called *Compton scattering*, first studied by Compton both theoretically and experimentally. The change in the energy (or wave number) of the photon can be calculated by applying the relativistic energy-momentum conservation law. If the initial and final wave numbers of the photon are $\vec{k}_0$ and $\vec{k}$ respectively, and the momentum of the electron after the collision is $\vec{p}$, energy-momentum conservation gives

$$\hbar\vec{k}_0 = \hbar\vec{k} + \vec{p};$$
$$c\hbar\,|\,\vec{k}_0\,| + mc^2 = c\hbar\,|\,\vec{k}\,| + \sqrt{m^2c^4 + |\,\vec{p}\,|^2} \tag{14}$$

On eliminating $\vec{p}$ between the two equations, we obtain

$$\frac{k}{k_0} = \frac{1}{1 + \frac{\hbar k_0}{mc}(1-\cos\theta)} \tag{15}$$

where θ is the scattering angle. The formula for scattering cross-section is also suitably modified; for *spinless* point particles it is

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \frac{1}{2}\left(\frac{k}{k_0}\right)^2\left(\frac{k_0}{k} + \frac{k}{k_0} - \sin^2\theta\right) \tag{16}$$

For k=k_0 this of course reduces to the Thomson formula. At $\theta = 0$, k=k_0 and there is no deviation from the Thomson result. As θ increases, k decreases; the maximum reduction occurs for $\theta = \pi$ where there is considerable reduction in the cross-section. For given θ the reduction is a function of the energy of the photon; the cross-section decreases with increasing energy.

We know that electrons are spin-half particles. For scattering by electrons the effect of the spin must also be taken into account which affects the results to some extent through the electron's magnetic moment.

29.2 Scattering by quasi-free charges

Experimentally the results for scattering of X-rays by atoms agree reasonably well with equation (12) at wide angles, particularly for light atoms. However, there is considerable deviation in the forward direction. The actual cross-section increases quite rapidly compared to what is dictated by the Thomson formula. The reason for this discrepancy is the coherent addition of amplitudes from all the electrons in the atom. Equations (1) and (2) now apply to each individual electron. If we assume, as we have already done in fact, that the amplitude of motion of the electrons is much smaller than the wavelength of the radiation and the distance of the point of observation is also much larger than the inter-electron distance, then the total electric field (radiation part only) due to a number of electrons will be

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\varepsilon_0 c}\sum_i e_i\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}}_i)}{R_i}\right]_{ret} \tag{17}$$

The acceleration for each electron is given by equation (4). Using this formula for the acceleration, we obtain

$$\vec{E}(\vec{x},t) = \frac{E_0}{c^2}\sum_j \frac{e_j^{\ 2}}{4\pi\varepsilon_0 m_j} \frac{\exp[i\vec{k}_0.\vec{x}_j - i\omega(t - \frac{R_j}{c})]}{R_j} \tag{18}$$

Since we are interested in radiation, distance R is large, and can be approximated by **[See Figure 14.8 from Jackson Edition 2; Same as in unit-28]**

$$R = x - \hat{n}.\hat{r} \tag{19}$$

Making this substitution in the above equation and going through the steps that led from equation (5) to equation (8) for the case of Thomson scattering by a free electron, we obtain

$$\frac{d\sigma}{d\Omega} = \left|\sum_j \frac{e^2}{4\pi\varepsilon_0 mc^2} e^{i\vec{q}.\vec{x}_j}\right|^2 \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \tag{20}$$

where

$$\vec{q} = \frac{\omega}{c}\hat{n} - \vec{k}_0 = \vec{k} - \vec{k}_0 \tag{21}$$

represents the change in the wave number vector.

Equation (20) applies to scattering by free charged particles instantaneously at position $\vec{x}_j$. But in an atom electrons are not free; they are bound to the nucleus. However, if the frequency of incident radiation is large compared to characteristic frequencies of binding, the electrons can be treated as almost free while being accelerated by a pulse of short duration. Thus equation (20) can be applied to scattering of radiation by even bound electrons as long as the frequency of the incident radiation is large compared to the binding frequencies of the electrons in the atom. Formula (20) applies to a specific position $\vec{x}_j$ for the j^{th} electron. We have to take the average of equation (20) over the position of all the particles in the atom (or the nucleus as the case may be) to get the cross-section. Thus

$$\frac{d\sigma}{d\Omega} = \left\langle \left|\sum_j \frac{e_j^{\ 2}}{4\pi\varepsilon_0 m_j c^2} e^{i\vec{q}.\vec{x}_j}\right|^2 \right\rangle \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \tag{22}$$

The symbol $\langle\ \rangle$ implies averaging over all possible values of $\vec{x}_j$.

The cross-section (22) depends crucially on the value of the "momentum transfer", $\vec{q}$. The quantity $\vec{x}_j$ has magnitude of the order of the dimensions of the bound system, say the atom. Let this linear size of the system be denoted by a. Then the cross-section behaves very differently for $qa \ll 1$ than for $qa \gg 1$. If θ is the scattering angle, then from equation (21)

$$q = 2k\sin(\frac{\theta}{2})\,. \tag{23}$$

Thus the dividing line between the two cases is at θ such that

$$2ka\sin(\frac{\theta}{2}) \sim 1 \tag{24}$$

In case the when the frequency is low so that *ka*<<1, the condition *qa*<<1 is satisfied for all angles. In case of high frequency, *ka*>>1, the condition *q a*<<1 will hold only for small angles. The dividing line between the two cases is for

$$\theta_{cr} \sim \frac{1}{ka}\,. \tag{25}$$

For angles much less than θ_{cr} the limit *q a*<<1 holds, whereas for angles much greater than θ_{cr}, the other limit *q a*>>1 holds.

Now the argument of the exponential factor in equation (22) is $\vec{q}.\vec{x}_j = qx_j\cos\alpha \lesssim qa$, and hence for *qa*<<1 the exponential factor can be approximated by unity. The exponential factor simplifies to

$$(\lim qa \to 0)\frac{d\sigma}{d\Omega} = \left\langle \left| \sum_j \frac{e_j^{\ 2}}{4\pi\varepsilon_0 m_j c^2} \right|^2 \right\rangle \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \tag{26}$$

For an atom of atomic number *Z*, $\sum_j \frac{e_j^{\ 2}}{4\pi\varepsilon_0 m_j c^2} = Z\frac{e^2}{4\pi\varepsilon_0 mc^2}$ and the cross-section simplifies to

$$(\lim qa \to 0)\frac{d\sigma}{d\Omega} = Z^2(\frac{e^2}{4\pi\varepsilon_0 mc^2})^2\left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \tag{27}$$

The factor Z^2 has appeared because of *coherence*. The amplitude due to all the electrons in the atom add coherently. Thus the total amplitude is *Z* times that due to one electron and the cross-section is Z^2 times.

In the opposite limit of *qa*>>1, the argument of the exponential factors are large and widely different in value. As a result when the average of the sum of squares of all the terms in equation (22) is taken, all the cross terms will vanish. Only the *Z* absolute square terms will survive and the cross-section becomes

$$(\lim qa \to \infty)\frac{d\sigma}{d\Omega} = \sum_j \left(\frac{e_j^{\ 2}}{4\pi\varepsilon_0 m_j c^2}\right)^2 \left|\hat{\varepsilon}^* . \hat{\varepsilon}_0\right|^2 \qquad (28)$$

As before, for the case of electrons in an atom the cross-section takes the form

$$(\lim qa \to \infty)\frac{d\sigma}{d\Omega} = Z\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^* . \hat{\varepsilon}_0\right|^2 \qquad (29)$$

This result corresponds to the incoherent addition of the amplitudes of all the electrons.

Let us estimate the critical angle, θ_{cr}, for the scattering of X-rays by atoms. In the Thomas-Fermi model of the atom, the atomic radius of an atom with atomic number *Z* is given by

$$a \simeq 1.4 a_0 Z^{-1/3} \qquad (30)$$

Here $a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{me^2}$ is the *Bohr radius* of the hydrogen atom. On substituting this expression into the definition (25) of the critical angle, we obtain

$$\theta_{cr} \sim \frac{1}{ka} \sim \frac{c}{\omega}\frac{me^2}{1.4 \times 4\pi\varepsilon_0 \hbar^2 Z^{-1/3}} = \frac{Z^{1/3}}{\hbar\omega}\left(\frac{me^2 c}{1.4(4\pi\varepsilon_0)\hbar}\right)$$

The expression in the parenthesis has the dimension of energy. If the numerical values of the various factors are substituted, this factor is nearly unity in units of keV. Hence

$$\theta_{cr} \sim \frac{Z^{1/3}}{\hbar\omega(keV)} \qquad (31)$$

For angles less than the critical angle, the cross-section rises rapidly to the value given by equation (27). At wide angles, the cross-section is given by equation (29), down by a factor of *Z* compared to (27).

29.3 Scattering by a bound charge

Let us now consider scattering of radiation by a bound charge. We assume that the electron is bound in the atom by a spherically symmetric linear restoring force along with a damping factor. The equation of motion of the electron is then

$$m(\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^{\ 2}\vec{x}) = e\vec{E} \qquad (32)$$

Here *m* is the mass of the electron, $m\omega_0^{\ 2}\vec{x}$ is the restoring force and $m\Gamma\dot{\vec{x}}$ is the damping force on the electron.

The origin of the damping force needs a little elaboration. Quantum mechanically, apart from scattering or re-emission of the radiation, other modes of decay are also allowed which is equivalent to a dissipation of the radiation. However, apart from this, the major source of dissipation is due to what is called *radiation reaction*. Proper understanding of this idea is difficult, it impinges on the realm of quantum electrodynamics, and in a sense it is not properly understood even within quantum electrodynamics, though otherwise it is a hugely successful theory. However the effect of radiation damping in many processes can be included in a simple manner and this is what we proceed to do here.

We have so far divided our study of electrodynamics into two categories and considered them independently. We have considered the production of electromagnetic field by moving charged particles along fixed trajectories, or the motion of charged particle in given electromagnetic fields. However, when a charge is placed in a given field, it is accelerated and produces radiation fields. These fields are bound to influence the subsequent motion of the charged particle. For a correct treatment of the problem this influence must be taken into account. This influence is the *radiation reaction* or *radiation damping*.

Let us see how we can take the effect of radiation damping in a heuristic way. The non-relativistic motion of a particle with charge e and mass m under the influence of an external force $\vec{F}_e$ is obtained from Newton's law

$$m\ddot{\vec{x}} = \vec{F}_e \tag{33}$$

To account for the effect of radiation on the motion of the particle, we modify this equation by including a reactive force terms, $\vec{F}_r$. So the equation now becomes

$$m\ddot{\vec{x}} = \vec{F}_e + \vec{F}_r \tag{34}$$

The question is: how to calculate this force? As we know (module 27), the power radiated by a non-relativistically moving charged particle undergoing acceleration $\vec{a} = \dot{\vec{v}}$ is given by (Larmor formula)

$$P = \frac{e^2 |\dot{\vec{v}}|^2}{6\pi\varepsilon_0 c^3} \tag{35}$$

We find the reactive force by using principle of conservation of energy: the negative of the work done by the force should equal the actual amount of energy radiated by the particle, say from time t_1 to t_2. In other words

$$\int_{t_1}^{t_2} \vec{F}_r.\vec{v}dt = -\frac{e^2}{6\pi\varepsilon_0 c^3}\int_{t_1}^{t_2} \dot{\vec{v}}.\dot{\vec{v}}dt \tag{36}$$

On doing the integral on the right by parts, we obtain

$$\int_{t_1}^{t_2} \vec{F}_r.\vec{v}dt = -\frac{e^2}{6\pi\varepsilon_0 c^3}\left[\vec{v}.\dot{\vec{v}}\right]_{t_1}^{t_2} + \frac{e^2}{6\pi\varepsilon_0 c^3}\int_{t_1}^{t_2} \vec{v}.\ddot{\vec{v}}dt \tag{37}$$

If the motion is periodic we take the interval (t_1, t_2) as one period. The state of the system will be the same at times t_1 and t_2. Or if the time interval is short, the state of the system will again be the same at times t_1 and t_2. In either case, the integrated term on the right hand side vanishes and we have

$$\int_{t_1}^{t_2} [\vec{F}_r - \frac{e^2}{6\pi\varepsilon_0 c^3}\ddot{\vec{v}}].\vec{v}dt = 0 \tag{38}$$

from which it follows that

$$\vec{F}_r = \frac{e^2}{6\pi\varepsilon_0 c^3}\ddot{\vec{v}} = m\tau\ddot{\vec{v}} \tag{39}$$

where

$$\tau = \frac{e^2}{6\pi\varepsilon_0 mc^3} \tag{40}$$

The modified equation (33) can therefore be put in the form

$$m\dot{\vec{v}} = \vec{F}_e + m\tau\ddot{\vec{v}}$$

Or

$$m(\dot{\vec{v}} - \tau\ddot{\vec{v}}) = \vec{F}_e \tag{41}$$

This modified equation of motion is known as *Abraham-Lorentz equation of motion*.

With the inclusion of the radiation reaction term via the Abraham-Lorentz equation, the equation of motion for a bound charge, equation (32) takes the form

$$m(\ddot{\vec{x}} - \tau\dddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0{}^2\vec{x}) = e\vec{E} \tag{42}$$

When the electric field is due to incident radiation, it takes the form

$$\vec{E}(\vec{x},t) = E_0\hat{\varepsilon}_0 e^{i(\vec{k}.\vec{x}-\omega t)} \tag{43}$$

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Here E_0 is the amplitude and $\hat{\varepsilon}$ the polarization vector of the electric field. If the field is weak, the amplitude of oscillation of the electron will be small and we can approximate the field by its value at $\vec{x} = 0$. Equation (42) then becomes

$$(\ddot{\vec{x}} - \tau\dddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^{\,2}\vec{x}) = \frac{e}{m}\hat{\varepsilon}_0\, E_0 e^{-i\omega t} \tag{44}$$

This has steady state solution of the type

$$\vec{x} = \hat{\varepsilon}_0 x_0 e^{-i\omega t} \tag{45}$$

On substituting this solution into equation (44), we obtain

$$\vec{x} = \frac{e}{m}\frac{E_0 e^{-i\omega t}}{\omega_0^{\,2} - i\,\tau\omega^3 - i\omega\Gamma - \omega^2}\hat{\varepsilon}_0 = \frac{e}{m}\frac{E_0 e^{-i\omega t}}{\omega_0^{\,2} - i\omega\Gamma_t - \omega^2}\hat{\varepsilon}_0 \tag{46}$$

where

$$\Gamma_t(\omega) = \Gamma + \omega^2\tau = \Gamma + (\frac{\omega}{\omega_0})^2(\omega_0^{\,2}\tau) = \Gamma + (\frac{\omega}{\omega_0})^2\Gamma' \tag{47}$$

$\Gamma_t(\omega)$ is called the *total decay constant*, and

$$\Gamma' = \omega_0^{\,2}\tau \tag{47a}$$

is the *radiative decay constant*. The acceleration of the charge is then

$$\ddot{\vec{x}} = -\frac{e}{m}\frac{\omega^2}{\omega_0^{\,2} - i\omega\Gamma_t - \omega^2}E_0 e^{-i\omega t}\hat{\varepsilon}_0 \tag{48}$$

Comparing it with equation (4) for Thomson scattering by a free electron we see that there is an additional factor of $(-\frac{\omega^2}{\omega_0^{\,2} - i\omega\Gamma_t - \omega^2})$ appearing in the expression for the acceleration. Going through exactly the same steps that led from equation (4) to equation (8) for the Thomson case yields the following formula for the scattering cross-section for the case of bound electron:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \frac{\omega^4}{(\omega_0^{\,2} - \omega^2)^2 + \omega^2\Gamma_t^{\,2}} \tag{49}$$

The scattering cross-section is thus the Thomson cross-section multiplied by the factor $\frac{\omega^4}{(\omega_0^{\,2} - \omega^2)^2 + \omega^2\Gamma_t^{\,2}}$.

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29.3.1 Rayleigh scattering and resonance fluorescence

Let us now analyze equation (49).

1. For frequencies small compared to the binding frequency, ($\omega << \omega_0$), the cross-section reduces to

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \left(\frac{\omega}{\omega_0}\right)^4 = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2 \left(\frac{\lambda_0}{\lambda}\right)^4$$

The scattering at small frequencies, or equivalently long wavelengths, is thus inversely proportional to the fourth power of the wavelength. This is the well-known *Rayleigh law of scattering*. This formula is also valid for the scattering of light by air molecules (nitrogen and oxygen). Air molecules scatter short wavelengths (blue) much more than the long wavelengths (red) thus leading to the blue colour of the sky. The ratio of the wavelength of red light is 650nm, that of the blue light is 450nm, so their ratio is 1.44. This makes scattering cross-section for blue light nearly eight times more than for red light. The redness of the rising and setting sun is also explained on the same basis.

2. Now consider the case of frequencies close to the resonant frequency, $\omega \approx \omega_0$. In this case we can make the following approximation in equation (49):

$$\frac{\omega^4}{(\omega_0{}^2 - \omega^2)^2 + \omega^2\Gamma_t{}^2} = \frac{\omega_0{}^4}{(\omega_0 + \omega)^2(\omega_0 - \omega)^2 + \omega_0{}^2\Gamma_t{}^2} \approx \frac{\Gamma'^2}{4\omega_0{}^2\tau^2[(\omega - \omega_0)^2 + (\Gamma_t/2)^2]}$$

where we have used equation (47a) in the last step. On using equation (40) for τ and this approximation in equation (49), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{9}{16}\left(\frac{c}{\omega_0}\right)^2 \frac{\Gamma'^2}{(\omega - \omega_0)^2 + (\Gamma_t/2)^2} \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2$$
$$= \frac{9}{64\pi^2}\lambda_0{}^2 \frac{\Gamma'^2}{(\omega - \omega_0)^2 + (\Gamma_t/2)^2} \left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2$$

As we have seen for the case of Thomson scattering, summing over the polarization of the scattered radiation, the factor $\left|\hat{\varepsilon}^*.\hat{\varepsilon}_0\right|^2$ is replaced by $\frac{1}{2}(1 + \cos^2\theta)$, and on further integrating on the entire solid angle it is replaced by $\frac{8\pi}{3}$. Hence, finally we get

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}\lambda_0{}^2 \frac{\Gamma'^2}{(\omega - \omega_0)^2 + (\Gamma_t/2)^2}$$

This is a typical resonance shape with half-width Γ_t and peak cross-section (at $\omega = \omega_0$) of

$$\sigma(\omega_0) = \frac{3}{2\pi}\lambda_0^{\ 2}(\frac{\Gamma'}{\Gamma_t})^2$$

This scattering having features of a sharp resonance is called *resonance fluorescence*. Fluorescence spectroscopy is used in, among others, biochemical, medical, and chemical research fields for analyzing organic compounds.

3. At very high frequencies, $\omega >> \omega_0$, the cross-section (49) approximates to

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^* . \hat{\varepsilon}_0\right|^2 \frac{\omega^4}{\omega^4 + \omega^2\Gamma_t^{\ 2}} \simeq \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\hat{\varepsilon}^* . \hat{\varepsilon}_0\right|^2 \frac{1}{1+\omega^2\tau^2}$$

since $\Gamma_t(\omega) = \Gamma + \omega^2\tau \rightarrow \omega^2\tau$. For high frequencies therefore, the cross-section tends to the Thomson formula except for the factor $(\frac{1}{1+\omega^2\tau^2})$ due to radiation damping. Using definition (40) of τ, $\omega\tau \approx 1$ implies $\omega\frac{e^2}{6\pi\varepsilon_0 mc^3} \approx 1$, or $\hbar\omega \approx 137mc^2$ on putting numerical values of various constants. Thus this factor becomes important only at extremely high energies, deep into the quantum domain. In other words, in the classical domain this factor can be totally neglected. The figure below the scattering cross-section (49) as a function of the angular frequency, ω. **[See Figure 17.3 from Jackson Edition 2]**

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Summary

1. *In this module we have considered the scattering of electromagnetic radiation by a free charged particle*
2. *Expression for the differential scattering cross-section as well as the total scattering cross-section, called Thomson formula is derived.*
3. *Cross-section for scattering by quasi-free system of charges is obtained. Conditions on the frequency and scattering angle for coherent and incoherent addition of the cross-section due to individual particles (electrons) are derived and the corresponding cross-sections obtained.*
4. *Expression for critical angle for the case of X-rays is obtained.*
5. *Next scattering by a bound charge is considered. The binding is provided by a harmonic restoring force. Resistive terms are also included in the equation of motion of the charged particle.*
6. *The idea of radiation resistance is explained briefly and the Abraham-Lorentz equation of motion derived.*
7. *Expression for the cross-section including the resistive forces is derived. The limiting cases of low, high and frequencies close to the binding frequency are discussed.*
8. *For low frequencies we are led to Rayleigh scattering. Rayleigh scattering, when applied to scattering by air particles explains the blue of the sky and redness of the morning and evening sun.*
9. *For frequencies close to the binding frequency it leads to the phenomenon of resonance fluorescence.*

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